Digital Logic Systems

Recitation 6: Representation of Boolean Functions by Formulas, Digital Abstraction

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April 11, 2019

Sum of Products

Recall the following definitions.

Definition

A variable or a negation of a variable is called a literal.

Definition

A formula that is the AND of literals is called a product term.

Definition

A simple product term p is a minterm with respect to a set U of variables if $vars(p) = U$.

A minterm is a simple product term, and therefore, every variable in U appears exactly once in p.

Minterms of a Boolean Function

Definition

For a $v \in \{0,1\}^n$, define the minterm p_v to be $p_v \triangleq (\ell_1^v \cdot \ell_2^v \cdots \ell_n^v)$, where:

$$\ell_i^v \triangleq \begin{cases} X_i & \text{if } v_i = 1 \\ \bar{X}_i & \text{if } v_i = 0. \end{cases}$$

Definition

Let $f^{-1}(1)$ denote the set

$$f^{-1}(1) \triangleq \{v \in \{0,1\}^n \mid f(v) = 1\}.$$

Definition

The set of minterms of f is defined by

$$M(f) \triangleq \{p_v \mid v \in f^{-1}(1)\}.$$

Sum of Products

Theorem

Every Boolean function $f : \{0,1\}^n \to \{0,1\}$ that is not a constant zero is represented by the sum of the minterms in $M(f)$.

Given f, generate the corresponding SOP formula

1. Find $f^{-1}(1)$ - all the input vectors $v \in \{0,1\}^n$ such that $f(v) = 1$
2. Convert the input vectors $v \in \{0,1\}^n$ into minterms $p = l_1 \cdot \ldots \cdot l_n$ by replacing $v_i = 0 \to l_i = \bar{X}_i$ and $v_i = 1 \to l_i = X_i$
3. SOP_f = The disjunction (OR-connective) of all the minterms p is the desired representation.

Sum of Products: Example

Example

Represent the following Boolean functions as a SOP formula:
(i)$f(a, b) = \max\{a, b\}$, (ii) $g(a, b) = \min\{a, b\}$.

Product of Sums

Theorem

Every Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ that is not a constant one can be represented by a product of maxterms.

Creating POS

1. $g \triangleq \bar{f}$
2. Find $g^{-1}(1)$ - all the input vectors $v \in \{0,1\}^n$ such that $g(v) = 1$
3. Convert the input vectors $v \in \{0,1\}^n$ into minterms $p = l_1 \cdot \ldots \cdot l_n$ by replacing $v_i = 0 \rightarrow l_i = \bar{X}_i$ and $v_i = 1 \rightarrow l_i = X_i$
4. SOP_g = The sum of all the minterms p is the desired representation.
5. $POS_f = \text{DM}(SOP_g)$

Product of Sums: Example

A similar "ritual"...

Example

Represent the following Boolean functions as an POS formula:
(i) $f(a,b) = \min\{a,b\}$, (ii) $h(a,b) = \max\{a,b\}$.

The finite Field $GF(2)$

Theorem

Every Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be represented by a polynomial in $GF(2)[U]$, where $U = \{X_1, \dots, X_n\}$.

Creating a polynomial for f

- easy: f is constant.
- $f^{-1}(1) \triangleq \{v \in \{0,1\}^n \mid f(v) = 1\}$.
- For each $v \in f^{-1}(1)$, we define the product p_v. The polynomial $p \in GF(2)[U]$ is defined as follows.

$$p \triangleq \bigoplus_{v \in f^{-1}(1)} p_v .$$

Properties

- $X \oplus X \Leftrightarrow 0$ $\quad X \oplus 1 \Leftrightarrow not(X)$ $\quad X \oplus 0 \Leftrightarrow X$
- $(X \oplus Y) \cdot Z \Leftrightarrow (X \cdot Z) \oplus (Y \cdot Z)$, for every $X, Y, Z \in \{0,1\}$.

$GF(2)$:Example

Example

Represent the Boolean function $carry : \{0,1\}^3 \to \{0,1\}$ by a polynomial in $GF(2)$.

$$carry(a,b,c) \triangleq \begin{cases} 1 & \text{if } a+b+c \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

a	b	c	carry(a,b,c)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$GF(2)$:Example

$\bar{a}bc + a\bar{b}c + ab\bar{c} + abc$

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$\Leftrightarrow (a \oplus 1)bc \oplus a(b \oplus 1)c \oplus ab(c \oplus 1) \oplus abc$

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$\Leftrightarrow 0 \oplus 0 \oplus bc \oplus ac \oplus ab$

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$GF(2)$:Question

Question

When can you swap the OR with XOR?

Example - SOP is not a compact representation

Example

Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function defined as follows.

$$f(X) \triangleq \begin{cases} 1 & \text{if } \forall 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \colon\ X_j = X_{n+1-j} \\ 0 & \text{otherwise.} \end{cases}$$

- Compute $M(f)$ for $n = 3$?

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  $\{(X_1 \cdot X_2 \cdot X_3), (X_1 \cdot \bar{X}_2 \cdot X_3), (\bar{X}_1 \cdot X_2 \cdot \bar{X}_3), (\bar{X}_1 \cdot \bar{X}_2 \cdot \bar{X}_3)\}$

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- Compute $M(f)$ for general n?
  $M(f) = \{(l_1 \cdot \ldots \cdot l_n)$ such that $\forall 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$: either l_j, l_{n+1-j} are both negated or both positive$\}$
  $|M(f)| = 2^{\lceil \frac{n}{2} \rceil}$

Example - a compact formula

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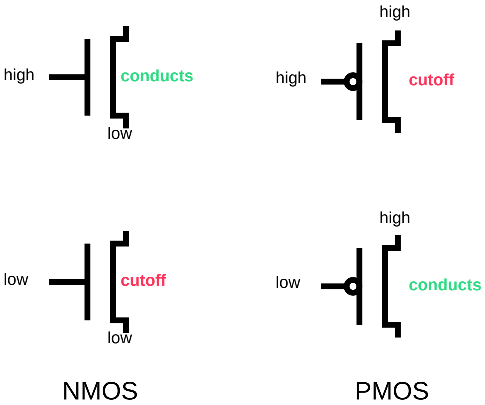
- Think of a $p \in GF(2)$
  $\prod_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} (X_j \oplus X_{n+1-j}) \oplus 1$
  Success: We dropped from an exponential number of terms to a linear number!

Digital Abstraction - Transistors

- The logical gates (AND, OR,...) are composed of transistors, which are analog devices that allow switching.
- Widely used MOSFET transistor has 2 types: N and P



- There are many other transistors (BJT, FinFet, JFET,...)

Digital Abstraction - CMOS technology

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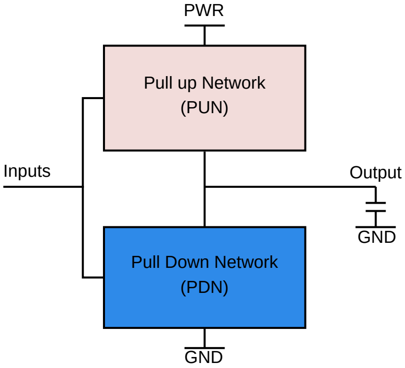

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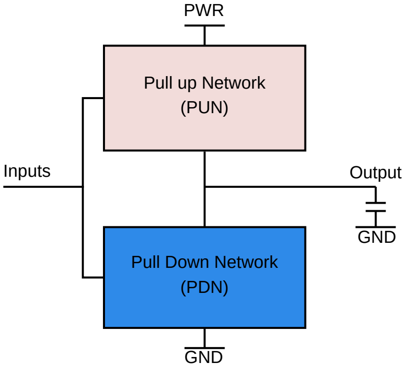

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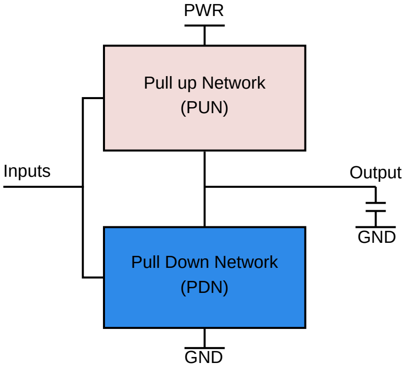
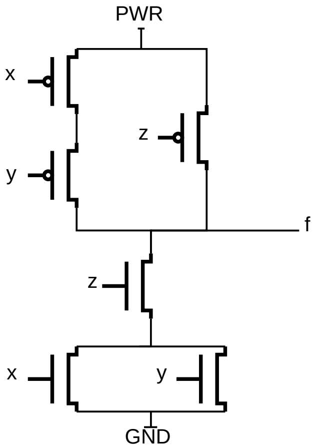
  1. PUN - contains only PMOS transistors. When conducting, pulls the output to high voltage (logical 1)
  2. PDN - contains only NMOS transistors. When conducting, pulls the output to low voltage (logical 0)
- PDN and PUN do not conduct at the same time (only for a short period of time, during the changes in the input)



Determine the Boolean function/formula implemented by the following CMOS circuit.



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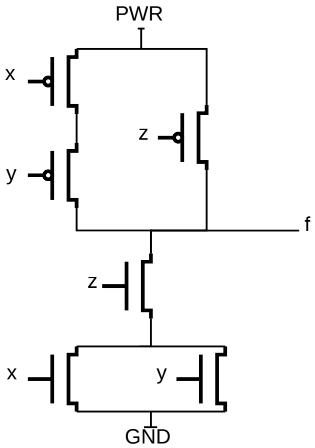

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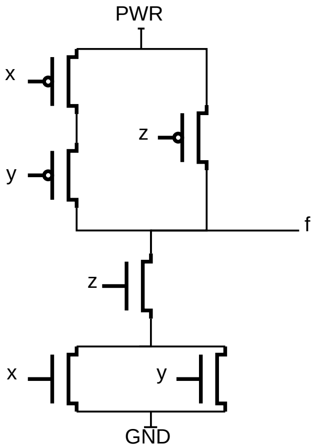

From PDN: $\bar{f} = (x \vee y) \wedge z$
From De-Morgan Dual: $f = DM(\bar{f}) = (\bar{x} \wedge \bar{y}) \vee \bar{z}$